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Symmetries of non-linear differential equations and linearisation

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Abstract. A non-linear ordinary differential equation is linearisable if it posseses SL(3, $\mathscr{R}$) symmetry. The conditions under which the Abelian two-dimensional subalgebras of sl(3, $\mathscr{R}$) are sufficient for linearisation are established.

1. Introduction

The Lie theory of extended groups has regained a lot of attention in recent studies of ordinary and partial differential equations. Knowing point symmetries of ordinary differential equations generally allows one to reduce the order of the equations (see, e.g., Bluman and Cole 1974, Ovsiannikov 1978). Alternatively, one can exploit the symmetries to construct first integrals for the given equations. When the equations are of Lagrangian type, this is most easily done in the context of Noether's theorem, but it has been pointed out by many authors that the point symmetries of Noether type do not exhaust the point symmetries of the differential equations (see, e.g., Lutzky 1978, Prince and Eliezer 1981, Wulfman and Wybourne 1976, Leach 1981). Concerning a single second-order equation, we know from Lie's counting theorem that there can be at most eight point symmetries (Ovsiannikov 1978, Anderson and Davison 1974). It is further known that all linear equations do have a full eight-parameter group of symmetries which is SL(3, $\mathscr{R}$). Mahomed and Leach (1985) recently added a new element to the discussion in investigating a non-linear equation of the type

$$\ddot{q}+\alpha q\dot{q}+\beta q^3=0 \tag{1}$$

which itself arose in a study of the generalised Emden equation (Leach 1985). They found that such an equation has only two point symmetries, unless $\alpha^2=9\beta$, in which case there are eight symmetries, whose generators again exhibit the sl(3, $\mathscr{R}$) commutation relations. The latter result means that there exists a *point transformation*, reducing the equation to any linear equation (such as the free particle equation) and that the non-linear equation accordingly can be readily solved. They further inferred from this example that any non-linear equation may have the SL(3, $\mathscr{R}$) group, provided it has eight symmetries. Certainly, in view of the invariance of the commutation relations of symmetry generators under arbitrary point transformations, having SL(3, $\mathscr{R}$) symmetry is a necessary and sufficient condition for a second-order differential equation

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to be linearisable (by which we mean linearisable through a point transformation). The question then arises whether we need to know the full eight symmetry generators of an equation before we can conclude that the linearisation will exist. The following property (cf Lie 1891) will be the starting point for the investigation of this paper.

Proposition. In order that a second-order ordinary differential equation has the $\mathrm{sl}(3, \mathscr{R})$ algebra, it is necessary and sufficient that it has the nilpotent algebra

$$[G_1, G_2]=0 \qquad [G_2, G_3]=0 \qquad [G_1, G_3]=G_2 \tag{2}$$

henceforth referred to as $\aleph$.

Proof. For a general second-order equation $\ddot{q}=f(\dot{q}, q, t)$, we write the generator of a point symmetry in the form

$$G=\tau(t, q)\partial/\partial t+\xi(t, q)\partial/\partial q. \tag{3}$$

Assuming that the equation has the algebra $\aleph$ then means that we have three symmetry generators G_1, G_2, G_3, satisfying the commutation relations (2).

If this is the case, we cannot have that both $G_1=\rho(q, t)G_2$ and $G_3=\psi(q, t)G_2$ (for suitable functions ρ and ψ) as the first two commutations then would imply $G_2(\rho)=0$, $G_2(\psi)=0$, which in turn would lead to $[G_1, G_3]=0$, contradicting the last relation in (2). Without loss of generality, we may therefore assume that no function ρ exists such that $G_1=\rho G_2$. As a result, there exists a regular point transformation $Q=Q(q, t)$, $T=T(q, t)$, transforming the generators G_1 and G_2 to the standard form

$$\bar{G}_1=\partial/\partial T \qquad \bar{G}_2=\partial/\partial Q. \tag{4}$$

Representing, in these coordinates, $\bar{G}_3$ in a general form like (3), it follows from the commutation relations involving G_3 that one must have

$$\partial\tau/\partial Q=0 \qquad \partial\xi/\partial Q=0 \qquad \partial\tau/\partial T=0 \qquad \partial\xi/\partial T=1.$$

Therefore (disregarding constant multiples of $\bar{G}_1$ and $\bar{G}_2$), the transformed equation must have the symmetry

$$\bar{G}_3=T\,\partial/\partial Q. \tag{5}$$

From (4) it is obvious that the right-hand side of the differential equation for Q cannot depend on T and Q. Expressing the invariance with respect to the generator (5) then straightforwardly shows that it cannot depend on $Q'=\mathrm{d}Q/\mathrm{d}T$ as well. The transformed equation will thus be of the form $Q''=\text{constant}$ (Lie 1891), which is an obviously linear differential equation and consequently has $\mathrm{SL}(3, \mathscr{R})$ symmetry. The original equation in q accordingly has the same symmetry, which completes the proof of the sufficiency. The necessity is trivial because $\aleph$ is a subalgebra of $\mathrm{sl}(3, \mathscr{R})$.

Remark. Whenever a non-linear differential equation is found to have eight symmetry generators, the above result also contains a hint for actually trying to construct a coordinate transformation which does the linearisation. Indeed, it will be advantageous to search for three combinations of the obtained generators which constitute $\aleph$. If this can be done, a linearisation will follow from transforming two of the commuting symmetries to the standard form (4).

One can push the question which has been dealt with in the above proposition a bit further and ask whether linearisability will not even follow from assuming less than the three-dimensional algebra $\aleph$. This is indeed the case. As a matter of fact, it follows from our argumentation that to within removing from G_3 a constant multiple of G_1, G_2 and G_3 commute and further satisfy $G_3 = \psi(t, q)G_2$ for some function ψ. This implies the existence of a coordinate transformation (already established by Lie (1891)) transforming the generators to

$$\bar{G}_2 = \partial/\partial Q \qquad \bar{G}_3 = T\,\partial/\partial Q \tag{6}$$

as a result of which the differential equation appears in the form $Q'' = F(T)$. Once again this is a linear equation having $\mathrm{SL}(3, \mathscr{R})$ symmetry.

The other subalgebra of (2) constitutes a more appealing subject for further study, if only because it is not trivial. Having two commuting symmetries which are independent means that the differential equation transforms to the general form (Lie 1891)

$$Q'' = F(Q') \tag{7}$$

which in itself is already a significant reduction, since equations of type (7) may be solved by two consecutive quadratures. In the following sections, we wish to investigate the possible existence of more symmetries for (7), thereby keeping an eye on the identification of further criteria which will ensure linearisability.

2. The symmetry conditions for a class of non-linear equations

Starting from the assumption that the equations under investigation have two independent commuting symmetries, we take the preliminary transformation to the standard form (4) of these symmetries for granted and rewrite the resulting equation in lower case variables as

$$\ddot{q} = f(\dot{q}). \tag{8}$$

A generator of the form (3) will be a symmetry for (8) if and only if τ and ξ satisfy the following requirement (invariance of (8) under the second extension of G):

$$\xi_{tt} + \dot{q}(2\xi_{tq} - \tau_{tt}) + \dot{q}^2(\xi_{qq} - 2\tau_{tq}) - \dot{q}^3\tau_{qq} + (\xi_q - 2\tau_t - 3\dot{q}\tau_q)f = [\xi_t + \dot{q}(\xi_q - \tau_t) - \dot{q}^2\tau_q]f_{\dot{q}} \tag{9}$$

where the suffices refer to partial derivatives.

It is impossible to proceed further with such a complicated equation without making further assumptions about the nature of the function $f(\dot{q})$. The polynomial character in $\dot{q}$ of all the coefficients in (9) strongly suggests looking at the case where f itself is a polynomial. Obviously we are not interested in the case that f is linear in $\dot{q}$. Suppose then that the leading order term in f is of degree n with $n \geqslant 3$. We write

$$f = k\dot{q}^n + l\dot{q}^{n-1} + \dots.$$

Looking at the coefficients of $\dot{q}^{n+1}$ and $\dot{q}^n$ in (9), we then obtain the following conditions:

$$(n-3)k\tau_q = 0 \tag{10}$$

$$(n-4)l\tau_q + k[(n-2)\tau_t - (n-1)\xi_q] = \tau_{qq}\delta_{n3} \tag{11}$$

where δ_{n3} is the Kronecker delta.

For any n greater than 3, it is clear that (10) and (11) allow τ at most to be a function of t and ξ to be a linear function in q with possible time-dependent coefficients. One can subsequently look at the terms of degree 0, 1 and 2 in (9) and infer from a simple analysis that in the most favourable case there can be at most three symmetry generators, provided the coefficients in f satisfy a number of algebraic relations. The bigger n is, the larger the number of such restrictions on the coefficients will be so that the chances of obtaining more than the two given symmetries become smaller with increasing n. In conclusion, linearisability is certainly not possible for a polynomial f of degree greater than three and the case $n=3$ appears to be of a peculiar nature, because it is the only case for which the coefficients of the highest-order terms in (9) automatically cancel out, leaving the possibility for a non-trivial q dependence in τ. For all these reasons, we further restrict ourselves to polynomials of degree 3. For completeness, we study the quadratic polynomial case in an appendix. The reader may find some similarities between our analysis and a recent paper by Aguirre and Krause (1985). These authors also treat the case of a polynomial of degree 3 (with coefficients possibly depending on q and t). However they merely compute the commutation relations of symmetry generators, without investigating the conditions under which such generators exist and without analysing the nature of the resulting algebra, nor the question of linearisabilty.

Considering a differential equation of the form

$$\ddot{q} = g\dot{q}^3 + a\dot{q}^2 + b\dot{q} + c \qquad g \neq 0 \tag{12}$$

where all coefficients are constant, we can make a preliminary rescaling of time to make the coefficient of the high-order term equal to one. Having done so, a transformation of the form

$$q = Q - \tfrac{1}{3}aT \qquad t = T \tag{13}$$

will eliminate the quadratic term on the right-hand side. So, without loss of generality, we may assume that $g=1$, $a=0$ in (12) and therefore restrict our attention to equations of the form

$$\ddot{q} = \dot{q}^3 + b\dot{q} + c. \tag{14}$$

Let us repeat first of all that an equation like (14) can certainly be solved by two quadratures. Being able to solve a differential equation does not necessarily mean, however, that the same equation can be linearised by a point transformation. Among other things, we wish to find out under what circumstances a linearisation for (14) exists, thereby keeping in mind that we regard (14) as a represenative of a whole class of differential equations which can be transformed to it once two commuting symmtries are known. It will appear soon that the study of the full symmetry group of (14) is quite interesting in its own right.

With an $f(\dot{q})$ as in (14), equating the coefficients of like powers of $\dot{q}$ in the symmetry condition (9) give rise to the following set of partial differential equations:

$$\tau_{qq} + 2\xi_q - \tau_t = 0 \tag{15}$$

$$2b\tau_q + 3\xi_t - \xi_{qq} + 2\tau_{tq} = 0 \tag{16}$$

$$b\tau_t + 3c\tau_q - 2\xi_{tq} + \tau_{tt} = 0 \tag{17}$$

$$b\xi_t - c\xi_q + 2c\tau_t - \xi_{tt} = 0. \tag{18}$$

We can solve (15) for ξ_q and use the result to solve (16) for ξ_t, thus obtaining

$$\xi_q = \tfrac{1}{2}(\tau_t - \tau_{qq}) \tag{19}$$

$$\xi_t = -\tfrac{1}{6}(\tau_{qqq} + 3\tau_{tq} + 4b\tau_q). \tag{20}$$

Substituting these expressions into (17) and (18) we obtain two partial differential equations involving τ alone. Expressing in addition that we want the system (19) and (20) to be completely integrable for ξ we end up with the following requirements on τ (because of the order of the equations,, it is preferable to abandon the index notation for partial derivatives at this stage):

$$\frac{\partial^3 \tau}{\partial t\, \partial q^2} + 3c\frac{\partial \tau}{\partial q} + b\frac{\partial \tau}{\partial t} = 0 \tag{21}$$

$$\frac{\partial^4 \tau}{\partial t\, \partial q^3} - b\frac{\partial^3 \tau}{\partial q^3} + 3\frac{\partial^3 \tau}{\partial t^2\, \partial q} + 3c\frac{\partial^2 \tau}{\partial q^2} + b\frac{\partial^2 \tau}{\partial t\, \partial q} - 4b^2\frac{\partial \tau}{\partial q} + 9c\frac{\partial \tau}{\partial t} = 0 \tag{22}$$

$$\frac{\partial^4 \tau}{\partial q^4} + 4b\frac{\partial^2 \tau}{\partial q^2} + 3\frac{\partial^2 \tau}{\partial t^2} = 0. \tag{23}$$

The strategy now is clear: we have to solve the above equations for τ first, after which (19) and (20) will determine the corresponding components ξ of symmetry generators. Before embarking upon this task, let us try to simplify the equations for τ. In the first place, using $\partial/\partial q$ of condition (21), (22) readily simplifies to

$$-b\frac{\partial^3 \tau}{\partial q^3} + 3\frac{\partial^3 \tau}{\partial t^2\, \partial q} - 4b^2\frac{\partial \tau}{\partial q} + 9c\frac{\partial \tau}{\partial \tau} = 0. \tag{24}$$

Next, observe that the combination $(21)_{qq}$-$(23)_t$ results in a homogeneous third-order partial differential equation for τ, namely

$$\frac{\partial^3 \tau}{\partial t^3} + b\frac{\partial^3 \tau}{\partial t \partial q^2} - c\frac{\partial^3 \tau}{\partial q^3} = 0. \tag{25}$$

In order to see whether (25) can actually replace the fourth-order equation (23), suppose conversely that τ is a solution of the set of equations (21), (24) and (25). Then, we can arrive at an equation of type (23) in two different ways. Indeed, it is straightfoward to verify that

$$(24)_q - 3(21)_t \sim b(23)$$

$$3(25)_q - (24)_t - 4b(21)_q \sim c(23).$$

Therefore we distinguish two cases.

Case I. b and c not both zero. It then follows from the above observations that the original conditions on τ can equivalently be replaced by the set of third-order conditions (21), (24) and (25).

Case II. $b = c = 0$. The requirements for τ readily simplify to

$$\frac{\partial^3 \tau}{\partial t\, \partial q^2} = 0 \qquad \frac{\partial^3 \tau}{\partial t^2\, \partial q} = 0 \qquad \frac{\partial^4 \tau}{\partial q^4} + 3\frac{\partial^2 \tau}{\partial t^2} = 0. \tag{26}$$

Remark. There is still a certain amount of redundancy hidden in the conditions for case I. For example, when $b=0$ ($c\neq 0$) one can show that, apart from the homogeneous equation (25), the other requirements integrate to the single condition $\tau_{,tq}+3c\tau=$ constant. On the other hand, for $b\neq 0$ we have seen above that (21) and (24) actually imply (23) and therefore also (25). It is, however, more convenient to study first the solutions of the homogeneous equation (25) anyway and in doing so it will turn out that a discussion about b being zero or not is rather irrelevant.

All partial differential equations for τ we have been dealing with have certain remarkable properties which can be of great help in constructing particular solutions. Consider for example equation (21) which we rewrite in the form

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\tau}{\partial q^2}+b\tau\right)+\frac{\partial}{\partial q}(3c\tau)=0.$$

This form implies that there must exist a function ϕ such that

$$\frac{\partial^2\tau}{\partial q^2}+b\tau=\frac{\partial\phi}{\partial q}\qquad -3c\tau=\frac{\partial\phi}{\partial t}. \tag{27}$$

One can easily verify that if τ is a particular solution of (21), then the function ϕ generated by (27) is another particular solution of the same equation. In other words, (27) is a so-called auto-Bäcklund transformation for (21) (Rogers and Shadwick 1982). The general property behind this observation is as follows. Consider two relations of the form

$$a\frac{\partial^{k+l}u}{\partial x^k\,\partial y^l}+cu=\frac{\partial v}{\partial y}\qquad b\frac{\partial^{i+j}u}{\partial x^i\,\partial y^j}+eu=\frac{\partial v}{\partial x} \tag{28}$$

where a, b, c and e are constants. Expressing the integrability condition $v_{xy}=v_{yx}$ results in a partial differential equation to be satisfied by u. Eliminating, on the other hand, the derivatives of u from (28) by calculating the combination

$$b\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j}\left(\frac{\partial v}{\partial y}\right)-a\frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l}\left(\frac{\partial v}{\partial x}\right)$$

and making use of (28) again to express everything in terms of v, results in exactly the same partial differential equation to be satisfied by v.

Returning to our equations for τ one can observe that there are actually different ways of rewriting some of them in a form like (28) and there is always one particular solution at hand, namely $\tau=$ constant. Through the process of (27), for example, this solution will generate an infinite number of polynomial in q and t solutions of equation (21), which of course will be drastically limited by the restrictions coming from the other equations. We will not use this technique here, because the general solution of our equations (21), (24), and (25) can be obtained in a straightforward manner.

3. Obtaining the full symmetry group

3.1. *Triple root case*

We start with the simple case II, for which we have to solve equations (26). The first of these implies that τ must be of the form

$$\tau=f(q)+h(t)q+g(t) \tag{29}$$

and the functions f, g, h are then bound to satisfy the additional restrictions

$$\ddot{h}=0 \qquad f^{(4)}+3\ddot{g}=0.$$

As a result, the general solution for τ is given by

$$\tau=c_1+c_2q+c_3q^2+c_4q^3+c_5t+c_6tq+c_7(q^4-4t^2). \tag{30}$$

There are seven arbitrary constants in this expression, which give rise to seven different symmetry generators, the ξ component of which is readily obtained from equations (19) and (20). In addition, there is always the solution

$$\tau=0 \qquad \xi=c_0$$

which makes a total of eight symmetries, the maximum one can expect. In table 1 the eight generators below ($c_i \to G_{i+1}$), together with their commutation relations in table 2.

We observe immediately that the generators G_1, G_2 and G_3 in the above table satisfy the $\aleph$ commutation relations (2) and hence we know from the proposition that we are dealing with SL(3, $\mathscr{R}$) symmetry.

For treating case I, we will first solve the homogeneous equation (25). The nature of the solutions of (25) depends on the way the partial differential operator factorises into the composition of three first-order operators. This in turn is directly related to the discussion of the nature of the roots of the cubic equation $x^3+bx+c=0$. From this point of view, it is clear that the situation we have just covered in fact corresponds to the case of a triple root for the cubic equation. We therefore next discuss the case of a double root and finally the case of three distinct roots.

3.2. Two equal factors for equation (25)

Since there is no x^2 term in the above-mentioned cubic equation, the sum of the roots must be zero. Accordingly, for the case under investigation, we write the factorisation

Table 1

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
τ	0	1	q	q^2	q^3	t	tq	q^4-4t^2
ξ	1	0	0	$-q$	$-\frac{3}{2}q^2-t$	$\frac{1}{2}q$	$\frac{1}{4}q^2-\frac{1}{2}t$	$-4qt-2q^3$

Table 2

$[G_i, G_j]$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	0	G_2	$2G_3-G_1$	$-3G_4$	$\frac{1}{2}G_1$	G_6	$4G_5$
G_2		0	0	0	$-G_1$	G_2	$G_3-\frac{1}{2}G_1$	$-8G_4$
G_3			0	G_3	$\frac{3}{2}G_4+G_6$	$\frac{1}{2}G_3$	$\frac{3}{4}G_4+\frac{1}{2}G_6$	$2G_5-4G_7$
G_4				0	$2G_7$	0	$\frac{1}{2}G_5$	0
G_5					0	$-\frac{1}{2}G_5$	$\frac{1}{4}G_8$	0
G_6						0	$\frac{1}{2}G_7$	G_8
G_7							0	0
G_8								0

of equation (25) in the form

$$\left(\frac{\partial}{\partial t}+2\alpha\frac{\partial}{\partial q}\right)\left(\frac{\partial}{\partial t}-\alpha\frac{\partial}{\partial q}\right)^2\tau=0 \tag{31}$$

with $\alpha\neq 0$. We then have

$$b=-3\alpha^2 \qquad c=-2\alpha^3. \tag{32}$$

The general solution of (31) is given by (compare with (29))

$$\tau=f(q-2\alpha t)+g(q+\alpha t)+(q-2\alpha t)h(q+\alpha t) \tag{33}$$

where f, g and h are as yet arbitrary functions of their arguments. According to the results of the previous section, we next have to turn to equations (21) and (24). Equation (21) becomes

$$-2\alpha f'''+\alpha(g'''-9\alpha^2 g')+\alpha(q-2\alpha t)(h'''-9\alpha^2 h')=0$$

where the prime denotes differentiation with respect to the appropriate argument. In view of the different arguments involved and the fact that $\alpha\neq 0$, it follows that we must have

$$f'''=0 \qquad g'''-9\alpha^2 g'=0 \qquad h'''-9\alpha^2 h'=0. \tag{34}$$

The remaining condition (24) then turns out to be identically satisfied. Solving equations (34) is a straightforward matter. The resulting solution for τ takes the form

$$\begin{aligned}\tau=c_1+c_2(q-2\alpha t)+c_3(q-2\alpha t)^2+c_4\sinh 3\alpha(q+\alpha t)\\ +c_5\cosh 3\alpha(q+\alpha t)+c_6(q-2\alpha t)\sinh 3\alpha(q+\alpha t)\\ +c_7(q-2\alpha t)\cosh 3\alpha(q+\alpha t).\end{aligned} \tag{35}$$

Just as in the previous case, this means that we again have a full eight-parameter group of symmetries, the generators of which are given in table 3. For shorthand, we introduce the new variables

$$T=q+\alpha t \qquad Q=q-2\alpha t. \tag{36}$$

We do not list the commutation relations of these generators. They turn out to be rather messy and it would be a major task to try and reorganise the set of generators in such a way that some standard sl(3, $\mathscr{R}$) pattern would emerge. The hints which were explored in the introduction appear to be of great help here for convincing

Table 3.

	τ	ξ
G_1	0	1
G_2	1	0
G_3	Q	$-\alpha Q$
G_4	Q^2	$-\alpha Q^2-Q$
G_5	$\sinh 3\alpha T$	$\frac{1}{2}\alpha\sinh 3\alpha T-\frac{3}{2}\alpha\cosh 3\alpha T$
G_6	$\cosh 3\alpha T$	$\frac{1}{2}\alpha\cosh 3\alpha T-\frac{3}{2}\alpha\sinh 3\alpha T$
G_7	$Q\sinh 3\alpha T$	$\frac{1}{2}(\alpha Q-1)\sinh 3\alpha T-\frac{1}{2}(3\alpha Q+1)\cosh 3\alpha T$
G_8	$Q\cosh 3\alpha T$	$-\frac{1}{2}(3\alpha Q+1)\sinh 3\alpha T+\frac{1}{2}(\alpha Q-1)\cosh 3\alpha T$

ourselves that we are indeed dealing again with SL(3, $\mathscr{R}$) symmetry. To see this, observe that we have

$$G_7 - G_8 = Q(G_5 - G_6) \tag{37}$$

and

$$(G_5 - G_6)(Q) = 0 \tag{38}$$

from which it follows that

$$[G_7 - G_8, G_5 - G_6] = 0. \tag{39}$$

The symmetries $G_7 - G_8$ and $G_5 - G_6$ therefore can play the role of G_2 and G_3 in the introductory discussion leading to equation (6) and their transformation to the standard form (6) will linearise the equation, meaning that the algebra of symmetries must be a representation of sl(3, $\mathscr{R}$).

3.3. *Three distinct factors for equation (25)*

In this case, the factorisation of (25) is

$$\left(\frac{\partial}{\partial t} + 2\alpha\frac{\partial}{\partial q}\right)\left(\frac{\partial}{\partial t} - (\alpha+\gamma)\frac{\partial}{\partial q}\right)\left(\frac{\partial}{\partial t} - (\alpha-\gamma)\frac{\partial}{\partial q}\right)\tau = 0. \tag{40}$$

The constant α thereby is real, whereas γ can be real or purely imaginary, depending on whether our related cubic equation has three distinct real roots or one real and two complex conjugate roots. In both cases we certainly have

$$\gamma \neq \pm 3\alpha \qquad \gamma \neq 0 \tag{41}$$

and the expression for the coefficients b and c in terms of α and γ is given by

$$b = -(3\alpha^2 + \gamma^2) \qquad c = -2\alpha(\alpha^2 - \gamma^2). \tag{42}$$

The general solution of (40) is

$$\tau = f(g - 2\alpha t) + g[q + (\alpha+\gamma)t] + h[q + (\alpha - \gamma)t]. \tag{43}$$

Inserting this expression into the remaining conditions (21) and (24) again gives rise to simple differential equations for f, g and h and it follows as before that there are eight symmetry generators. For writing them down in a real form, one has to make a distinction between the case γ real and the case γ purely imaginary. Since the full expressions become rather complicated, we prefer to continue the probe into the nature of the algebra after a simplifying transformation.

3.3.1. *Three real roots.*

The factorisation (40) of the partial differential equation (25) means that we are actually looking at the case for which our differential equation (14) can be written as

$$\ddot{q} = (\dot{q} - 2\alpha)[\dot{q} + (\alpha+\gamma)][\dot{q} + (\alpha - \gamma)]. \tag{44}$$

Under the coordinate transformation

$$\bar{q} = q + (\alpha+\gamma)t \qquad \bar{t} = q + (\alpha - \gamma)t \tag{45}$$

equation (44) reduces to the form

$$\bar{q}'' = (3\alpha - \gamma)\bar{q}'^2 - (3\alpha + \gamma)\bar{q}'. \tag{46}$$

So we are down to an equation of the type discussed in the appendix. We can further reduce it to one of the standard types treated there by the subsequent transformation

$$\tilde{q} = (3\alpha - \gamma)\bar{q} - \tfrac{1}{2}(3\alpha - \gamma)\bar{t} \qquad \tilde{t} = \bar{t}. \tag{47}$$

This brings us to the equation

$$\tilde{q}'' = \tilde{q}'^2 - \tfrac{1}{4}(3\alpha + \gamma)^2 \tag{48}$$

which clearly is of type (A2). Since we found SL(3, $\mathscr{R}$) symmetry for (A2), we can draw the same conclusion about (44) and the symmetry generators, if wanted, can be obtained from those in table A1, using the transformations (45) and (47).

3.3.2. One real and two complex conjugate roots. The factorisation (40), with $\gamma = \mathrm{i}\omega$, now relates to a differential equation of the form

$$\ddot{q} = (\dot{q} - 2\alpha)(\dot{q}^2 + 2\alpha\dot{q} + \alpha^2 + \omega^2). \tag{49}$$

The transformation

$$\bar{q} = -(9\alpha^2 + \omega^2)t \qquad \bar{t} = q - 2\alpha t \tag{50}$$

is found to reduce the cubic right-hand side (49) again to a quadratic one, explicitly

$$\bar{q}'' = \bar{q}'^2 - 6\alpha\bar{q}' + 9\alpha^2 + \omega^2. \tag{51}$$

Putting

$$\tilde{q} = \bar{q} - 3\alpha\bar{t} \qquad \tilde{t} = \bar{t} \tag{52}$$

eventually brings us to the standard type

$$\tilde{q}'' = \tilde{q}'^2 + \omega^2 \tag{53}$$

for which the symmetry generators are listed in table A1. Again, we have SL(3, $\mathscr{R}$) symmetry.

Let us briefly discuss the meaning of the results of this section. When one performs an arbitrary coordinate transformation

$$T = F(q, t) \qquad Q = G(q, t) \tag{54}$$

on the free particle equation $Q'' = 0$, there results a differential equation which is at most cubic in $\dot{q}$. Explicity it becomes

$$\begin{aligned}[FG]_{t,q}\ddot{q} + [F, G]_{q,q^2}\dot{q}^3 + ([F, G]_{t,q^2} + 2[F, G]_{q,tq})\dot{q}^2 \\ + ([F, G]_{q,t^2} + 2[F, G]_{t,tq})\dot{q} + [F, G]_{t,t^2} = 0\end{aligned} \tag{55}$$

where, for example, $[F, G]_{q,q^2}$ is a shorthand notation for

$$[F, G]_{q,q^2} = \frac{\partial F}{\partial q}\frac{\partial^2 G}{\partial q^2} - \frac{\partial^2 F}{\partial q^2}\frac{\partial G}{\partial q}.$$

Equation (55) therefore represents the most general type of second-order differential equation which can be linearised by a point transformation. It is not at all obvious from this observation that all constant coefficient equations like (12) are linearisable, because this involves imposing four partial differential equations on the two functions F and G. Our results, however, show that indeed all equations of type (12) are linearisable. Moreover, we started looking at such equations as representing a whole class of equations which have two commuting symmetries. In this respect, we are now able to state the following conclusion: a second-order differential equation which has two commuting point symmetries G_1, G_2 (G_2 not being of the form $\rho(q, t)G_1$ for some function ρ) is linearisable, if and only if the transformation which brings G_1, G_2 into standard form (4) reduces the differential equation to one which is at most cubic in $\dot{q}$.

To complete this study, we construct in the next section a linearising transformation for all cases we have been led to distinguish.

4. Linearising transformations

4.1. The triple root case

For the equation

$$\ddot{q} = \dot{q}^3 \tag{56}$$

we observed in § 3 that the algebra $\{G_1, G_2, G_3\}$ is $\aleph$. Now, even though the commuting generators G_1 and G_2 are in standard form, our equation (56) is not linear! This can only mean that we have, so to speak, the wrong coordinate playing the role of t. The transformation

$$Q = t \qquad T = q \tag{57}$$

indeed reduces the equation to $Q'' = -1$, so that the relation

$$Q = -\tfrac{1}{2}T^2 + c_1 T + c_2 \tag{58}$$

through (57) implicitly defines the solution of (56).

4.2. The double root case

For a differential equation of the form

$$\ddot{q} = (\dot{q} - 2\alpha)(\dot{q} + \alpha)^2 \tag{59}$$

we know from the discussion in the previous section that a linearisation can be obtained if we transform the symmetry generators $G_5 - G_6$ and $G_7 - G_8$ to the form (6). It turns out, however, that the coordinates (36), which were merely introduced to simplify the notation, do the job as well. They reduce equation (59) to the linear equation

$$Q'' = 3\alpha Q' \tag{60}$$

whose solution is given by

$$Q = c_1 + c_2\, e^{3\alpha T}. \tag{61}$$

The solution of (59) thus is implicitly defined by (61) and the transformation formulae (36).

4.3. The case of three distinct real roots

In view of the reduction which was achieved in the preceding section, we merely have to linearise equation (A2). From the observations in the appendix, it follows that a linearisation will be obtained if we transform for example the symmetries G_7 and G_3 to the standard form $\partial/\partial T$, $\partial/\partial Q$. Such a transformation must satisfy the requirements

$$\begin{aligned} &e^{q-\omega t}\left(\frac{\partial T}{\partial q}\frac{\partial}{\partial T}+\frac{\partial Q}{\partial q}\frac{\partial}{\partial Q}\right)=\frac{\partial}{\partial T} \\ &e^{-2\omega t}\left(\frac{\partial T}{\partial t}\frac{\partial}{\partial T}+\frac{\partial Q}{\partial t}\frac{\partial}{\partial Q}\right)+\omega\, e^{-2\omega t}\left(\frac{\partial T}{\partial q}\frac{\partial}{\partial T}+\frac{\partial Q}{\nabla q}\frac{\partial}{\partial Q}\right)=\frac{\partial}{\partial Q} \end{aligned} \tag{62}$$

or

$$\frac{\partial Q}{\nabla q}=0 \qquad e^{-2\omega t}\frac{\partial Q}{\partial t}=1$$

$$e^{q-\omega t}\frac{\partial T}{\partial q}=1 \qquad \frac{\partial T}{\partial t}+\omega\frac{\partial T}{\partial q}=0.$$

We straighforwardly obtain

$$Q=\frac{1}{2\omega}e^{2\omega t} \qquad T=-e^{-q+\omega t} \tag{63}$$

and the reduced differential equation becomes $Q''=0$. Hence, the relation

$$c_1\, e^{-q-\omega t}+c_2\, e^{-2\omega t}=1 \tag{64}$$

implicitly defines the solution of (A2). For obtaining the solution of (44), we just have to replace ω by $\frac{1}{2}(3\alpha+\gamma)$ and take account of the two linear transformations (45) and (47).

4.4. The case of one real and two complex conjugate roots

For linearising equation (A4), we can, for example, exploit the fact that the generators G_7 and G_8 commute and are proportional to each other. Therefore, a transformation which reduces G_7 to $\partial/\partial Q$ and G_8 to $T\,\partial/\partial Q$ will do the job. Such a transformation is easily obtained as

$$T=\tan\omega t \qquad Q=-e^{-q}/\cos\omega t \tag{65}$$

and the reduced equation happens to be the free particle equation. As a result, the relation

$$(c_1\sin\omega t+c_2\cos\omega t)\,e^{q}=1 \tag{66}$$

defines the solution of (A4) and obtaining the solution of (49) is just a matter of taking further account of the linear transformations (50) and (52).

5. Discussion

The general problem of equivalence for second-order equations $\ddot{q}=H(\dot{q},q,t)$ under point transformation was considered by Tresse (1896). He constructed all semi-invariants for such equations. For our discussion, however, it suffices to mention only the functionally independent order-four semi-invariants of which there are two. The vanishing of one of them $\partial^4 H/\partial\dot{q}^4$ is a necessary condition for linearisation (i.e.

reduction to any linear equation such as, for example the free particle equation). This, one may recall, also follows from equation (55). Furthermore, for an equation in normal form (Arnold 1983)

$$\ddot{q} = A(t)q^2 + \mathrm{O}(|q|^3 + |\dot{q}|^3)$$

the other semi-invariant is the scalar invariant

$$I = 5A\ddot{A} - 12\dot{A}^2$$

which is constructed from the differential form of order $\frac{5}{2}$, namely

$$\omega = A(t)|\mathrm{d}t|^{5/2}.$$

It turns out that the vanishing of the form ω is a necessary condition for linearisation. The two conditions $\omega = 0$ and $\partial^4 H/\partial\dot{q}^4 = 0$, however, are sufficient for linearisation. The geometric theory of second-order equations, therefore, also provides criteria for linearisation. In this regard we further cite the work of Cartan (1924) who in particular investigated second-order equations cubic in the first derivative. The question of linearisability of differential equations has recently been discussed by other authors as well. We can mention, for example, a paper by Berkovich (1979) on ordinary differential equations, quadratic in $\dot{q}$ and a contribution by Kumei and Bluman (1982), dealing primarily with systems of partial differential equations. The ideas put forward in our paper bear some resemblance to the latter reference, because we discuss linearisability in terms of the existence of certain point symmetries of the differential equation.

We have shown that every equation which has two commuting (non-proportional) symmetries will be linearisable, provided that the transformation which brings these symmetries in their standard form reduces the differential equation to one which is at most cubic in $\dot{q}$. An interesting topic for further study would be to find practical criteria for testing given second-order differential equations with respect to the existence of two such commuting point symmetries.

We have also observed that, when the equation has two proportional commuting symmetries, linearisation is immediate. In this way, we have in fact treated two of the four possible cases in the classification of two-dimensional algebras of symmetry generators of a single second-order differential equation (Lie 1891). The other two cases are characterised by

$$[G_1, G_2] = G_1 \qquad \text{with} \qquad G_2 \neq \rho(q, t)G_1 \tag{67}$$

or

$$[G_1, G_2] = G_1 \qquad \text{with} \qquad G_2 = \rho(g, t)G_1. \tag{68}$$

In the latter case, the standard form for the generators is given by $\bar{G}_1 = \partial/\partial Q$, $\bar{G}_2 = Q\partial/\partial Q$ and the transformed differential equation is linear in Q' (Lie 1891). Whenever an equation has two point symmetries (68), we can therefore again conclude that it will have $\mathrm{SL}(3, \mathscr{R})$ symmetry. Just as it was the case with two commuting symmetries, the situation is less trivial when G_1 and G_2 are not proportional, as in (67). The standard form of such symmetries then is given by

$$\bar{G}_1 = \frac{\partial}{\partial Q} \qquad \bar{G}_2 = T\frac{\partial}{\partial T} + Q\frac{\partial}{\partial Q}$$

and the transformed differential equation is $Q''=T^{-1}F(Q')$ (Lie 1891). The study, along the lines of the present paper, of the possible existence of more than two point symmetries for the differential equation corresponding to this case is currently under investigation.

Finally, we would like to point out that one can think of generalising this study to systems with more than one degree of freedom, whereby the generalisation of the case treated in detail here would be the most appealing one. We would then be talking about systems

$$\ddot{q}^i = f^i(t, q, \dot{q}) \qquad i=1,\ldots,n$$

having up to $n+1$ commuting point symmetries, which accordingly can be transformed to the standard form $\partial/\partial T$, $\partial/\partial Q^1, \ldots, \partial/\partial Q^n$.

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Appendix. Quadratic in $\dot{q}$ equations

Considering a general equation of the form

$$\ddot{q} = a\dot{q}^2 + b\dot{q}^1 + c \tag{A1}$$

we can perform a preliminary rescaling of q to make $a=1$. Having done so, a transformation of the form $Q=q+\frac{1}{2}bt$ will further eliminate the $\dot{q}$ term. Henceforth, we can restrict our attention to the case where $a=1$ and $b=0$ in (A1) and we have to distinguish two different situations.

A1. The right-hand side has real roots

The equation under investigation has the form

$$\ddot{q} = \dot{q}^2 - \omega^2. \tag{A2}$$

The general symmetry requirement (9) for this case gives rise to the following set of partial differential equations:

$$\begin{aligned}
&\tau_q + \tau_{qq} = 0 \\
&\xi_q - \xi_{qq} + 2\tau_{tq} = 0 \\
&2\xi_t - 3\omega^2\tau_q - 2\xi_{tq} + \tau_{tt} = 0 \\
&\omega^2\xi_a - 2\omega^2\tau_t - \xi_{tt} = 0.
\end{aligned} \tag{A3}$$

They can easily be solved and are found to produce the eight symmetry generators given in table A1.

Table A1

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
τ	0	1	$e^{-2\omega t}$	$e^{2\omega t}$	$e^{-q+\omega t}$	$e^{-q-\omega t}$	0	0
ξ	1	0	$\omega\, e^{-2\omega t}$	$-\omega\, e^{2\omega t}$	$-\omega\, e^{-q+\omega t}$	$\omega\, e^{-q-\omega t}$	$e^{q-\omega t}$	$e^{q+\omega t}$

Their commutation relations are given in table 5. It is clear from this table that, for example, $\{G_3, G_6, G_7\}$ constitute the $\aleph$ algebra, from which it follows that the symmetry group of equation (A2) is SL(3, $\mathcal{R}$). We recall that the linearising transformation is given by (63), thereby reducing (A2) to the free particle equation.

A2. The right-hand side has complex roots

We are now talking about an equation of the form

$$\ddot{q} = \dot{q}^2 + \omega^2. \tag{A4}$$

Table A2

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	0	0	0	$-G_5$	$-G_6$	G_7	G_8
G_2		0	$-2\omega G_3$	$2\omega G_4$	ωG_5	$-\omega G_6$	$-\omega G_6$	ωG_8
G_3			0	$4\omega G_2$	$2\omega G_6$	0	0	$2\omega G_7$
G_4				0	0	$-2\omega G_5$	$-2\omega G_8$	0
G_5					0	0	$G_2-3\omega G_1$	G_4
G_6						0	G_3	$G_2+3\omega G_1$
G_7							0	0
G_8								0

Table A3

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
τ	0	1	$\cos 2\omega t$	$\sin 2\omega t$	$e^{-q}\sin \omega t$	$e^{-q}\cos \omega'$	0	0
ξ	1	0	$\omega \sin 2\omega t$	$-\omega \cos 2\omega t$	$-\omega\, e^{-q}\cos \omega t$	$\omega\, e^{-q}\sin \omega t$	$e^{q}\cos \omega t$	$e^{q}\sin \omega t$

Table A4

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	0	0	0	$-G_5$	$-G_6$	G_7	G_8
G_1		0	$-2\omega G_4$	$2\omega G_3$	ωG_6	$-\omega G_5$	$-\omega G_8$	ωG_7
G_3			0	$2\omega G_2$	ωG_8	ωG_5	ωG_8	ωG_7
G_4				0	ωG_5	$-\omega G_6$	$-\omega G_7$	ωG_8
G_5					0	0	$\frac{1}{2}G_4-\frac{3}{2}\omega G_1$	$\frac{1}{2}G_2-\frac{1}{2}G_3$
G_6						0	$\frac{1}{2}G_2+\frac{1}{2}G_3$	$\frac{1}{2}G_4+\frac{3}{2}G_1$
G_7							0	0
G_8								0

The partial differential equations for obtaining all point symmetries of (A4) of course differ only slightly from (A3). We again find eight generators which are listed in table A3. The commutator algebra is given in table A4. The $\aleph$ subalgebra is found with the generators $\{G_8, G_7, \frac{1}{2}(G_2+G_3)\}$ or $\{G_5, G_6, \frac{1}{2}(G_2+G_3)\}$. So it is by now no longer a surprise that we again have SL(3, $\mathscr{R}$) symmetry. The linearisation of (A4) via point transformation is given by (65). Once more the transformed equation is the free particle equation.

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